FYUGP BSc 1st Semester

Classical Algebra (Unit 2)

Algebric Equations:

Relation between Roots and Coefficients .

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General Form of a Polynomial

Consider a general polynomial of degree n:

$$P(x)=a_n x^n+a_{n-1}x^{n-1}+\ldots\ldots\ldots\ldots\ldots+a_1 x+a_0$$

where $a_n, a_{n-1}, \ldots\ldots\ldots, a_0$ are the coefficients and $a_n \neq 0$.

If $r_1, r_2, \ldots\ldots\ldots, r_n$ are the roots of this polynomial, then it can also be expressed in factored form as:

$$P(x)=a_n(x-r_1)(x-r_2)\ldots\ldots\ldots\ldots(x-r_n).$$

1. Quadratic Polynomial (Degree 2)

For a quadratic polynomial

$$ax^2+bx+c=0,$$

with roots α and β, the relationships are:

1. **Sum of the roots**: $\alpha+\beta=-b/a$
2. **Product of the roots**: $\alpha\beta=c/a$

Example: Solve the polynomial $2x^2+5x-3=0$

Here, **$a = 2$, $b = 5$, $c = -3$.**

- Sum of roots: **$\alpha+\beta= -5/2$**
- Product of roots: **$\alpha\beta= -3/2$**

Solving these we get ,

the roots of this equation are **$x=1/2$** and **$x=-3$.**

- Sum: $1/2+(-3)=-5/2$
- Product: $(1/2)(-3)=-3/2$
- The calculated values match the relationships.

2. Relation for Cubic Equation:

For $ax^3+bx^2+cx+d=0$,

Let the roots are **α,β,γ.**

Relations are:

1. **Sum of roots:**

   $\alpha+\beta+\gamma = -b/a$

2. **Sum of product of roots two at a time:**

   $\alpha\beta+\beta\gamma+\gamma\alpha = c/a$

3. **Product of roots:**

   $\alpha\beta\gamma = -d/a$

Example: **Solve the equation:** $x^3-6x^2+11x-6=0$.

Here **a=1, b=−6, c=11, d=−6.**

- Sum of roots = −(-6)/1=6
- Sum of products two at a time = 11/1=11
- Product of roots = −(-6)/1=6
- The roots are **1,2,3.**

Check: **1+2+3=6,**

1·2+2·3+3·1=11,

1·2·3=6.

3. Relation for Quartic Equation:

For $ax^4+bx^3+cx^2+dx+e=0$,

let the roots be $\alpha,\beta,\gamma,\delta$.

Relations are:

1. $\alpha+\beta+\gamma+\delta=-b/a$
2. $\alpha\beta+\beta\gamma+\gamma\delta+\alpha\gamma+\alpha\delta+\beta\delta=c/a$
3. $\alpha\beta\gamma+\beta\gamma\delta+\gamma\delta\alpha+\delta\alpha\beta=--d/a$
4. $\alpha\beta\gamma\delta =e/a$

4. General Rule (For any degree n) *(Vieta's Formula)*

For polynomial

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$$

roots $\alpha_1, \alpha_2, \ldots, \alpha_n$ satisfy:

- **Sum of roots (one at a time):**

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n = -\, a_{n-1} / a_n$$

- **Sum of products of roots taken two at a time:**

$$\sum_{1 \le i < j \le n} \alpha_i \alpha_j = a_{n-2} / a_n$$

- **Sum of products taken three at a time:**

$$\sum_{1 \le i < j < k \le n} \alpha_i \alpha_j \alpha_k = -a_{n-3} / a_n.$$

… continues until

- **Product of all roots:**

$$\alpha_1 \alpha_2 \ldots\ldots\ldots \alpha_n = (-1)^n \, a_0 / a_n$$

In general, the sum of the products of the roots taken k at a time is given by:

$$\sum_{i1<\ldots<ik} r_{i1}\ldots\ldots\ldots r_{ik} = (-1)^k a_{n-k} / a_n$$

- **In short:** **Coefficients ↔ Roots** are connected by alternating signs and ratios of coefficients.
- This powerful set of formulas is a key component of polynomial theory. It provides a bridge between the **algebraic structure** of a polynomial (its coefficients) and the **geometric properties** of its roots.

Example 1: Quartic Polynomial (Degree 4)

Solve the polynomial $x^4-10x^3+35x^2-50x+24=0$.

The coefficients are $a_4=1$, $a_3=-10$, $a_2=35$, $a_1=-50$, $a_0=24$.

Let the roots be r_1, r_2, r_3, r_4.

Applying formulas we get that

- **Sum of the roots**: $r_1+r_2+r_3+r_4=-a_3/a_4=-(-10)/1=10$
- **Sum of the products of roots taken two at a time**:

$$r_1r_2+r_1r_3+r_1r_4+r_2r_3+r_2r_4+r_3r_4=a_2/a_4=35/1=35$$

- **Sum of the products of roots taken three at a time**:

$$r_1r_2r_3+r_1r_2r_4+r_1r_3r_4+r_2r_3r_4=-a_1/a_4=-(-50)/1=50$$

- **Product of all roots**: $r_1r_2r_3r_4=a_0/a_4=24/1=24$

Solving the above 4 relations we get the roots of this given polynomial are 1,2,3,4.

Verification :

- **Sum**: 1+2+3+4=10
- **Sum of products (two time)**:
  (1)(2)+(1)(3)+(1)(4)+(2)(3)+(2)(4)+(3)(4)=2+3+4+6+8+12=35
- **Sum of products (three at a time)**:
  (1)(2)(3)+(1)(2)(4)+(1)(3)(4)+(2)(3)(4)=6+8+12+24=50
- **Product**: (1)(2)(3)(4)=24

Hence verified.

Example 2: Using the formulas to construct a polynomial.

Construct a cubic polynomial with roots 2,−1, and 3.

Solution: The general form of a cubic polynomial is

x^3−(sum of roots)x^2+(sum of products of roots taken two at a time)x−(product of all roots)=0.

Let the roots be $r_1 = 2$, $r_2 = -1$, $r_3 = 3$.

1. **Sum of the roots**: $2+(-1)+3=4$
2. **Sum of the products of roots taken two at a time**:

   $(2)(-1)+(-1)(3)+(3)(2)=-2-3+6=1$

1. **Product of all roots**: $(2)(-1)(3)=-6$

Substituting these values into the general form, we get:

$$x^3-(4)x^2+(1)x-(-6)=0$$

i.e. $$x^3-4x^2+x+6=0$$

This is the polynomial with the given roots.

Assignment : Answer the followings

Short Type Questions :

1. For a quadratic equation $ax^2+bx+c=0$, what is the sum of its roots?
2. The roots of the cubic equation $2x^3-5x^2+x+7=0$ are **α,β, and γ**.
   What is the value of the product $\alpha\beta\gamma$?
3. If the sum of the roots of the quadratic equation $3x^2+kx-5=0$ is **4**,
   what is the value of **k**?
4. The roots of the quadratic equation $2x^2+5x-1=0$ are **α** and **β**.
   Find the value of $1/\alpha+1/\beta$.
5. For the polynomial $P(x)=x^4-2x^3+3x^2-4x+5=0$, what is the sum of the products of its roots taken two at a time?

Long Type :

1 Given the cubic equation $x^3+ax^2+bx+c=0$, if the roots are **1,2, and 5**, find the values of a, b, and c.

2 For the polynomial $x^3-8x^2+kx-12=0$, the roots are in an arithmetic progression. Find the value of **k**.

3 A quartic equation $x^4+ax^3+bx^2+cx+d=0$ has roots $2+i, 2-i, 1$, **and** 3. Find the value of **$a+b$.**

4 If the sum of two roots of the cubic polynomial $x^3-9x^2+kx-15=0$ is **5**, find the roots of the polynomial.

5 A **cubic** polynomial has roots **1,−2, and 3**. Find the equation of the polynomial?

Practice makes a man perfect.

Practice……..Practice ……..and ……….Practice is the way of life.

Maths(God) bless You.